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On the non-hereditary recursion operator and the constraint on the potential associated with the Giachetti–Johnson equation and its gauge equivalent Yang equation

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Abstract. The recursion operator associated with the GJ equation is shown not to be hereditary. Restricting the potential to the invariant subspace of the recursion operator leads to a constraint on the potential. Under the constraint two systems obtained from the GJ equation and the related time evolution equation for eigenfunctions are shown to be naturally consistent. Constants of the motion for former system are given and a solution to this system satisfies a certain higher-order stationary equation. Also, similar results are obtained for the Yang equation.

1. Introduction and notation

The central role in studying an integrable equation in $1+1$ dimensions is played by the hereditary recursion operator (see, for example, [1-5]) which satisfies some algebraic geometrical properties mentioned in [5]. In this paper we study some properties of the non-hereditary recursion operator.

Consider the generalised AKNS eigenvalue problem

$$\varphi_x = M\varphi \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad M = \begin{pmatrix} -\lambda + w & u \\ v & \lambda - w \end{pmatrix} \tag{1.1}$$

which is proposed by Giachetti and Johnson in [6] and is called the GJ equation for short, and its gauge equivalent Yang equation [7]

$$\psi_x = \tilde{M}\psi \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \tilde{M} = \begin{pmatrix} s & \xi + q + r \\ -\xi - q + r & -s \end{pmatrix} \tag{1.2}$$

where λ and ξ are eigenparameters, and u, v, w, s, q, r are sufficiently smooth functions of x and t. Equation (1.1) is the special case of the spectral problem considered in [8]. It was pointed out in [8, 9] that the general form of the nonlinear evolution equations associated with (1.1) consists of a term expressed in recursion form by a recursion operator and an additional arbitrary term. By suitably choosing the arbitrary function, an infinite set of heirarchies of nonlinear evolution equations associated with (1.1) and (1.2), for which the two terms mentioned above can be rewritten in one term

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with another recursion operator, are given in [9], respectively, as follows:

$$p_t = \theta L^{*n} f_0 \tag{1.3a}$$

$$p_t = \theta_l L^{*n+1} f_0 \qquad l = 1, 2, \ldots \tag{1.3b}$$

$$\bar{p}_t = \bar{\theta} \bar{L}^{*n} \bar{f}_0 \tag{1.4a}$$

$$\bar{p}_t = \bar{\theta}_l \bar{L}^{*n+1} \bar{f}_0 \qquad l = 1, 2, \ldots \tag{1.4b}$$

where

$$p = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \bar{p} = \begin{pmatrix} q \\ r \\ s \end{pmatrix} \qquad f_0 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \qquad \bar{f}_0 = \begin{pmatrix} 2\mathrm{i} \\ 0 \\ 0 \end{pmatrix}$$

$$\theta = \begin{pmatrix} 0 & D-2w & 0 \\ D+2w & 0 & 0 \\ 0 & 0 & -\frac{1}{2}D \end{pmatrix} \qquad \bar{\theta} = \begin{pmatrix} \frac{1}{2}D & 0 & 0 \\ 0 & \frac{1}{2}D & q \\ 0 & -q & \frac{1}{2}D \end{pmatrix} \qquad D = \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.5}$$

$$\theta_l = \begin{pmatrix} 0 & -2 & lu \\ 2 & 0 & -lv \\ -lu & lv & lD \end{pmatrix} \qquad \bar{\theta}_l = \begin{pmatrix} -D & -ls & lr \\ ls & 0 & -\mathrm{i} \\ -lr & \mathrm{i} & 0 \end{pmatrix} \qquad l = 1, 2, \ldots$$

the operators L^* and $\bar{L}^*$ are adjoint operators of L and $\bar{L}$, respectively. L^* and $\bar{L}^*$ are given by

$$L^* = \begin{pmatrix} \frac{1}{2}D + w & 0 & -\frac{1}{2}v \\ 0 & -\frac{1}{2}D + w & -\frac{1}{2}u \\ D^{-1}u(D+2w) & D^{-1}v(D-2w) & 0 \end{pmatrix} \tag{1.6a}$$

$$\bar{L}^* = \begin{pmatrix} 0 & D^{-1}(2sq - rD) & D^{-1}(-2qr - sD) \\ -r & -q & \frac{1}{2}D \\ -s & -\frac{1}{2}D & -q \end{pmatrix} \tag{1.6b}$$

$$D^{-1}D = DD^{-} = 1.$$

It is known [6, 8] that (1.1) can be converted to the canonical ZS-AKNS spectral problem by the gauge transformation $q = u\mathrm{e}^{-2D^{-1}w}$, $r = v\mathrm{e}^{2D^{-1}w}$, which transforms the triple of functions (u, v, w) to the pair (q, r) in the ZS-AKNS case. It is clear that this gauge transformation admits the uncertainty of the general nonlinear evolution equations for (u, v, w) when starting from the evolution equations for (q, r). Thus, after specifying the uncertainty, it is significant to study the specific equations (1.3) and (1.4) in themselves. Indeed, we find that the recursion operator L for (1.3) possesses some properties which are quite different from those for the hereditary recursion operator associated with the AKNS hierarchy [10]. The main reason for this difference is that L is not hereditary. By using the results given in [2, 11], we shall show that L is not hereditary.

It is significant to consider a constraint on the potential of an eigenvalue problem and associated integrable nonlinear evolution equations (see, for example, [12, 14] and the references within [13]). If no boundary condition for the potential is required, we proposed in [13] a straightforward way of obtaining the constraint on the potential by restricting a hierarchy of integrable evolution equations to the invariant subspace of their recursion operator. Under this constraint condition, two systems obtained from the Lax pair can be shown to be naturally consistent. Usually, the invariant subspace of a hereditary recursion operator consists of the eigenvectors of the recursion operator. However, in section 3, for the non-hereditary recursion operator L, we have to restrict p to a subspace spanned by eigenvectors of L^* and one more vector $\Psi_0=(0,0,2)^{\mathrm{T}}$ in order to obtain an invariant subspace of L^* and a constraint on p. Under this constraint on p, two systems obtained from the GJ equation (1.1) and time evolution equation of φ related to (1.3) are shown to be naturally consistent. Also, the constants of the motion for the former system are given and the solution of this system satisfies a certain higher-order stationary equation of (1.3). Using the gauge transformation, it is easy to obtain the constraint on q and r in the AKNS case from one on u, v and w. However, it seems that this gauge transformation does not provide a direct way of obtaining the specific constraint on u, v and w from one on q and r. Similar results for the Yang equation are also obtained.

2. Non-hereditary property of L and $\bar{L}$

It was shown in [9] that (1.3b) can be written as a Hamiltonian systems with θ_l as the associated symplectic operator, namely $p_t=\theta_l L^{*n+1}f_0=\theta_l\delta I_{n+1}/\delta p$. But it is easy to verify that $\theta_l L^*\neq L\theta_l$. This means [11] that L is not hereditary. (Otherwise, it would be valid that $\theta_l L^*=L\theta_l$.) This conclusion is consistent with the fact that for (1.3a), $L^{*n}f_0=\delta I_n/\delta p$, $\theta L^*=L\theta$, but θ is not a symplectic operator.

Also, we can show that L is not hereditary by using other property of hereditary. From (1.6a) we have

$$L=\begin{pmatrix} -\frac{1}{2}D+w & 0 & (D-2w)uD^{-1} \\ 0 & \frac{1}{2}D+w & (D+2w)vD^{-1} \\ -\frac{1}{2}v & -\frac{1}{2}u & 0 \end{pmatrix}.$$

Taking

$$\kappa_0=\begin{pmatrix} 2u \\ -2v \\ 0 \end{pmatrix}$$

it is easy to check that

$$L'[k_0]=\begin{pmatrix} 0 & 0 & (D-2w)(2u)D^{-1} \\ 0 & 0 & (D+2w)(-2v)D^{-1} \\ v & -u & 0 \end{pmatrix}$$

$$k_0'L=\begin{pmatrix} -D+2w & 0 & 2(D+2w)uD^{-1} \\ 0 & -D-2w & -2(D+2w)vD^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$Lk_0'=\begin{pmatrix} -D+2w & 0 & 0 \\ 0 & -D-2w & 0 \\ -v & u & 0 \end{pmatrix}.$$

Thus we have

$$L'[k_0]=k_0'L-Lk_0'$$

which means that L is a strong symmetry for k_0 (see [2]).

Setting

$$k_1 = Lk_0 = \begin{pmatrix} -u_x + 2uw \\ -v_x - 2vw \\ 0 \end{pmatrix}$$

then a straightforward calculation gives

$$L'[k_1] = \begin{pmatrix} 0 & 0 & (D-2w)(-u_x+2uw)D^{-1} \\ 0 & 0 & (D+2w)(-v_x-2vw)D^{-1} \\ \frac{1}{2}v_x + vw & \frac{1}{2}u_x - uw & 0 \end{pmatrix}$$

$$k_1'L = \begin{pmatrix} \frac{1}{2}(-D+2w)^2 - uv & -u^2 & -(D-2w)^2uD^{-1} \\ v^2 & -\frac{1}{2}(D+2w)^2 + uv & -(D+2w)^2vD^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$Lk_1' = \begin{pmatrix} \frac{1}{2}(-D+2w)^2 & 0 & (-\frac{1}{2}D+w)(2u) \\ 0 & -\frac{1}{2}(D+2w)^2 & (\frac{1}{2}D+w)(-2v) \\ -\frac{1}{2}v(-D+2w) & \frac{1}{2}u(D+2w) & 0 \end{pmatrix}.$$

It follows from the above formulae that

$$L'[k_1] - k_1'L + Lk_1' \neq 0$$

which means that L is not a strong symmetry for k_1. Since L is a strong symmetry for k_0, if L is hereditary, it would follow that L is a strong symmetry for $k_1 = Lk_0$. So L is not hereditary [2].

The spectral problem (1.2) can be converted to (1.1) by a gauge transformation given in [9]. It is shown in [9] that

$$\bar{L} = TLT^{-1} \qquad T = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ \frac{1}{2}\mathrm{i} & -\frac{1}{2}\mathrm{i} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

which means that $\bar{L}$ can never be hereditary [2].

Finally, we want to point out that L^* and $\bar{L}^*$ satisfy the following isospectal eigenvalue equations:

$$L^* G_\lambda = \lambda G_\lambda \tag{2.1}$$

$$\bar{L}^* G_\xi = \xi G_\xi \tag{2.2}$$

where G_λ and G_ξ denote the gradients of λ and ξ, respectively. Assume that u, v, w, q, r, s belong to Schwartz space. Using the following formula [15] that if

$$\varphi_x = M\varphi \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

then

$$\int_{-\infty}^{\infty} (-\dot{c}\varphi_1^2 + 2\dot{a}\varphi_1\varphi_2 + \dot{b}\varphi_2^2)\, \mathrm{d}x = 0$$

where the dot denotes the Frechet derivative, it is easy to find the gradients G_λ and G_ξ from (1.1) and (1.2), respectively. We have

$$G_\lambda = \left(\frac{\delta\lambda}{\delta u}, \frac{\delta\lambda}{\delta v}, \frac{\delta\lambda}{\delta w}\right)^{\mathrm{T}} = (\varphi_2^2, -\varphi_1^2, 2\varphi_1\varphi_2)^{\mathrm{T}}$$

$$G_\xi = \left(\frac{\delta\xi}{\delta q}, \frac{\delta\xi}{\delta r}, \frac{\delta\xi}{\delta s}\right)^{\mathrm{T}} = (-\psi_1^2 - \psi_2^2, \psi_1^2 - \psi_2^2, -2\psi_1\psi_2)^{\mathrm{T}}.$$

From (1.1), one gets

$$(\varphi_1^2)_x = 2(-\lambda + w)\varphi_1^2 + 2u\varphi_1\varphi_2 \qquad (\varphi_2^2)_x = 2v\varphi_1\varphi_2 - 2(-\lambda + w)\varphi_2^2$$

$$(\varphi_1\varphi_2)_x = u\varphi_2^2 + v\varphi_1^2$$

which can be rewritten as

$$\lambda\varphi_2^2 = \tfrac{1}{2}(\varphi_2^2)_x + w\varphi_2^2 - v\varphi_1\varphi_2 \tag{2.3}$$

$$-\lambda\varphi_1^2 = \tfrac{1}{2}(\varphi_1^2)_x - w\varphi_1^2 - u\varphi_1\varphi_2 \tag{2.4}$$

$$\begin{aligned} 2\lambda\varphi_1\varphi_2 &= D^{-1}(2\lambda u\varphi_2^2 + 2\lambda v\varphi_1^2) \\ &= D^{-1}(u(\varphi_2^2)_x - 2uv\varphi_1\varphi_2 + wu\varphi_2^2 - v(\varphi_1^2)_x + 2wv\varphi_1^2 + 2uv\varphi_1\varphi_2) \\ &= D^{-1}[u(D+2w)\varphi_2^2 - v(D-2w)\varphi_1^2]. \end{aligned} \tag{2.5}$$

Equations (2.3), (2.4) and (2.5) admit the equation (2.1). Similarly, (2.2) can be deduced from (1.2).

Remark. We have just shown that L is not a strong symmetry for $p_t = k_1$, which is the first equation in the hierarchy (1.3a). Thus the formula (2.1) does not contradict the conclusion that L is not hereditary (see [11]).

3. The natural constraint on p and $\bar{p}$

Besides (1.1), if φ satisfies

$$\varphi_t = N\varphi \qquad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{3.1}$$

where

$$A = \sum_{k=0}^{n-1} a_k\lambda^{n-k} \qquad B = \sum_{k=1}^{n} b_k\lambda^{n-k} \qquad C = \sum_{k=1}^{n} c_k\lambda^{n-k} \tag{3.2}$$

$$\begin{pmatrix} c_k \\ b_k \\ 2a_k \end{pmatrix} = L^{*k}f_0 \qquad k = 0, \ldots, n \tag{3.3}$$

then the solvability condition of (1.1) and (3.1) is

$$\varphi_{xt} - \varphi_{tx} = M_t - N_x + MN - NM = 0. \tag{3.4}$$

Using (3.2) and (3.3), we have [9]

$$M_t - N_x + MN - NM = \begin{pmatrix} w_t + a_{nx} & u_t - (b_{nx} - 2wb_n) \\ v_t - (c_{nx} + 2wc_n) & -w_t - a_{nx} \end{pmatrix} \tag{3.5}$$

which, together with (3.4), gives (1.3a). Here (1.3a) is deduced from (1.1) and (3.1) without requiring any boundary condition for p, and we define the integral constant of D^{-1} appearing in L^* to be zero. Using $I_0=\int_{x_0}^{x} \mathrm{d}y$ instead of D^{-1}, we define

$$L_0=\begin{pmatrix} \frac{1}{2}D+w & 0 & -\frac{1}{2}v \\ 0 & -\frac{1}{2}D+w & -\frac{1}{2}u \\ I_0u(D+2w) & I_0v(D-2w) & 0 \end{pmatrix}$$

where x_0 is a fixed arbitrary constant. It is easy to see from (2.1) that if φ satisfies (1.1), then

$$L_0\Psi=\lambda\Psi+e\Psi_0 \tag{3.6}$$

where

$$\Psi=\begin{pmatrix} \varphi_2^2 \\ -\varphi_1^2 \\ 2\varphi_1\varphi_2 \end{pmatrix} \qquad \Psi_0=\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \qquad e=-\lambda\varphi_1\varphi_2|_{x=x_0}.$$

We now consider following system instead of (1.1):

$$\begin{aligned} &\Phi_{jx}=M_j\Phi_j && \Phi_j=\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix} \\ &M_j=\begin{pmatrix} -\lambda_j+w & u \\ v & \lambda_j-w \end{pmatrix} && j=1,\ldots,N \end{aligned} \tag{3.7}$$

where $\lambda_k\neq\lambda_l$, when $k\neq l$. We define

$$\Phi=(\varphi_{11},\ldots,\varphi_{1N};\varphi_{21},\ldots,\varphi_{2N})^{\mathrm{T}}$$

$$\Psi_j=\begin{pmatrix} \varphi_{2j}^2 \\ -\varphi_{1j}^2 \\ 2\varphi_{1j}\varphi_{2j} \end{pmatrix} \qquad j=1,\ldots,N.$$

If Φ solves (3.7), one gets from (3.6)

$$L_0\Psi_j=\lambda_j\Psi_j+e_j\Psi_0 \tag{3.8}$$

where

$$e_j=-\lambda_j\varphi_{1j}\varphi_{2j}|_{x=x_0}. \tag{3.9}$$

Note that

$$L_0\Psi_0=f_1=\begin{pmatrix} -v \\ -u \\ 0 \end{pmatrix}. \tag{3.10a}$$

We can get an invariant subspace H of L_0 spanned by $\{\Psi_1,\ldots,\Psi_N,\Psi_0\}$ by demanding

$$\begin{pmatrix} -v \\ -u \\ 0 \end{pmatrix}=\sum_{j=0}^{N}\alpha_j\Psi_j.$$

Without loss of generality, we take $\alpha_j=1$, i.e.

$$f_1=\begin{pmatrix}-v\\-u\\0\end{pmatrix}=\sum_{j=1}^{N}\Psi_j+\Psi_0 \tag{3.10b}$$

which is equivalent to

$$v=-\sum_j \varphi_{2j}^2 \tag{3.11a}$$

$$u=\sum_j \varphi_{1j}^2 \tag{3.11b}$$

$$\sum_j \varphi_{1j}\varphi_{2j}=-1 \tag{3.11c}$$

where we use Σ_j instead of $\Sigma_{j=1}^N$ for brevity throughout the paper. Under the constraint condition (3.11a) and (3.11b), it follows from (3.7) that

$$\left(\sum_j \varphi_{1j}\varphi_{2j}\right)_x=0$$

which is consistent with (3.11c). Finally we have from (3.7) that

$$\begin{pmatrix}v\\u\\w\end{pmatrix}=f(\Phi)=\begin{pmatrix}-\Sigma_j\varphi_{2j}^2\\ \Sigma_j\varphi_{1j}^2\\ -\Sigma_j\lambda_j\varphi_{1j}\varphi_{2j}-\Sigma_j\varphi_{1jx}\varphi_{2j}+\Sigma_j\varphi_{1j}^2\,\Sigma_k\varphi_{2k}^2\end{pmatrix}. \tag{3.12}$$

Imposing the constraint condition (3.11c) and (3.12), (3.7) and (3.1) become

$$\Phi_{jx}=\bar{M}_j\Phi_j \qquad \bar{M}_j=M_j|_A \qquad j=1,\ldots,N \tag{3.13}$$

$$\Phi_{jt}=\bar{N}_j\Phi_j \qquad \bar{N}_j=N|_{\lambda=\lambda_j,A} \qquad j=1,\ldots,N \tag{3.14}$$

where subscript A means to substitute (3.11c) and (3.12) into the expression. We will show that (3.13) and (3.14) are naturally consistent, namely that the set of the solutions to (3.13) is left invariant under the flow (3.14). Indeed, the fact that the constraint on p (3.12) ensures that H be an invariant subspace of L_0 allows us to show the consistency of (3.13) and (3.14).

Let Φ satisfy (3.14); set

$$F_j=\Phi_{jx}-\bar{M}_j\Phi_j \qquad F_j=\begin{pmatrix}F_{1j}\\F_{2j}\end{pmatrix} \qquad F=(F_{11},\ldots,F_{1N};F_{21},\ldots,F_{2N}). \tag{3.15}$$

Following the procedure proposed in [13, 15], we will show that if $F(x,0)=0$, then $F(x,t)=0$.

Lemma. F defined by (3.15) satisfies

$$F_{jt}=\bar{N}_jF_j-\begin{pmatrix}w_t+a_{nx} & u_t-(b_{nx}-2wb_n)\\ v_t-(c_{nx}+2wc_n) & -w_t-a_{nx}\end{pmatrix}\Phi_j|_A \qquad j=1,\ldots,N. \tag{3.16}$$

Proof.

$$\begin{aligned}F_{jt}&=\Phi_{jxt}-\bar{M}_{jt}\Phi_j-\bar{M}_j\Phi_{jt}\\&=(N_{jx}\Phi_j+N_jM_j\Phi_j+N_jF_j-M_{jt}\Phi_j-M_jN_j\Phi_j)|_A\\&=\bar{N}_jF_j-(M_{jt}-N_{jx}+M_jN_j-N_jM_j)\Phi_j|_A\end{aligned}$$

which completes the proof by using (3.5). □

Theorem 3.1. Suppose that $\Phi(x,t)$ solves (3.14) and $\Phi(x,0)$ solves (3.13), then $\Phi(x,t)$ satisfies (3.13), and (3.12) is a solution to (1.3a).

Proof. Using (3.15), a straightforward calculation gives

$$L_0\Psi_j = \lambda_j\Psi_j + e_j\Psi_0 + G_j \qquad j = 1, \ldots, N \tag{3.17a}$$

where

$$e_j = -\lambda_j\varphi_{1j}\varphi_{2j}|_{x=x_0}$$

$$G_j = \begin{pmatrix} G_{1j} \\ G_{2j} \\ G_{3j} \end{pmatrix} = \begin{pmatrix} \varphi_{2j}F_{2j} \\ \varphi_{1j}F_{1j} \\ 2\lambda_j I_0(F_{1j}\varphi_{2j} + F_{2j}\varphi_{1j}) + 2I_0(u\varphi_{2j}F_{2j} - v\varphi_{1j}F_{1j}) \end{pmatrix}. \tag{3.17b}$$

From (3.17) and (3.10), it can be derived by induction that

$$L_0 f_1|_A = \sum_j (\lambda_j + 1)\Psi_j + \alpha_2\Psi_0 + G^{(1)} \tag{3.18}$$

$$L_0^k f_1|_A = \sum_j \Psi_j \sum_{m=0}^{k} \alpha_m\lambda_j^{k-m} + \alpha_{k+1}\Psi_0 + G^{(k)} \tag{3.19}$$

where

$$\alpha_0 = 1 \qquad \alpha_m = \sum_{l=0}^{m-1} \alpha_l\delta_{m-1-l} \tag{3.20}$$

$$\delta_0 = 1 \qquad \delta_l = \sum_j \lambda_j^{l-1}e_i = -\sum_j \lambda_j^l\varphi_{1j}\varphi_{2j}|_{x=x_0} \tag{3.21}$$

$$G^{(1)} = \sum_j G_j|_A \qquad G^{(k)} = \sum_j G_j \sum_{m=0}^{k-1} \alpha_m\lambda_j^{k-1-m} + L_0G^{(k-1)}|_A.$$

It is easy to observe that each monomial of each component of $G^{(k)}$ contains either one component of F, F_x, $F_{xx}, \ldots$ as a factor or an integration like $I_0 \ldots I_0 g$, the integrand of which, $g(x)$, has one component of F, $F_x, \ldots$ as a factor. In what follows, for simplicity, we denote all these kinds of function vector by the one symbol $\tilde{G} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3)^{\mathrm{T}}$ without regard for their differences.

Using the following identity repeatedly:

$$L^*\begin{pmatrix} c_k \\ b_k \\ 2a_k \end{pmatrix} = L_0\begin{pmatrix} c_k \\ b_k \\ 2a_k \end{pmatrix} + \beta_{k+1}\Psi_0 \qquad \beta_k = a_k|_{x=x_0} \tag{3.22}$$

we get from (3.3) and (3.19)

$$\begin{aligned} \left.\begin{pmatrix} c_{k+1} \\ b_{k+1} \\ 2a_{k+1} \end{pmatrix}\right|_A &= L^{*k}f_1|_A = \sum_{m=0}^{k} \beta_{k-m}L_0^m f_1|_A + \beta_{k+1}\Psi_0 \\ &= \sum_{m=0}^{k} \beta_{k-m} \sum_j \Psi_j \sum_{i=0}^{m} \alpha_i\lambda_j^{m-i} + \sum_{m=0}^{k+1} \alpha_m\beta_{k+1-m}\Psi_0 + \tilde{G} \\ &= \sum_j \Psi_j \sum_{m=0}^{k} h_m\lambda_j^{k-m} + h_{k+1}\Psi_0 + \tilde{G} \end{aligned} \tag{3.23}$$

where

$$h_0 = 1 \qquad h_m = \sum_{i=0}^{m} \alpha_i\beta_{m-i}|_A = \sum_{i=0}^{m} \alpha_i a_{m-i}|_{A,x=x_0}. \tag{3.24}$$

Using the identity

$$\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l(\lambda_m^{n-k}\lambda_j^{k-l-1}-\lambda_j^{n-k}\lambda_m^{k-l-1})=0 \tag{3.25}$$

one gets from (3.12), (3.14) and (3.23) that

$$\begin{aligned}
v_t|_A &= -2\sum_m \varphi_{2m}\varphi_{2mt}|_A \\
&= -2\sum_m \varphi_{2m}\left(\sum_{k=1}^{n} a_k\lambda_m^{n-k}\varphi_{1m}-\sum_{k=0}^{n-1} a_k\lambda_m^{n-k}\varphi_{2m}\right)\Big|_A \\
&= -2\sum_{m,j}\varphi_{1m}\varphi_{2m}\varphi_{2j}^2\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l(\lambda_m^{n-k}\lambda_j^{k-l-1}-\lambda_m^{k-l-1}\lambda_j^{n-k}) \\
&\quad +2\sum_m \varphi_{2m}^2\sum_{k=0}^{n-1} h_k\lambda_m^{n-k}-2\sum_{m,j}\varphi_{2m}^2\varphi_{1j}\varphi_{2j}\sum_{l=0}^{n-1} h_l\lambda_j^{n-l-1}+\tilde{G}_1 \\
&= 2\sum_m \varphi_{2m}^2\sum_{k=0}^{n-1} h_k\lambda_m^{n-k}-2\sum_{m,j}\varphi_{2m}^2\varphi_{1j}\varphi_{2j}\sum_{l=0}^{n-1} h_l\lambda_j^{n-l-1}+\tilde{G}_1 .
\end{aligned} \tag{3.26}$$

By using (3.15) and (3.23), it is found that

$$\begin{aligned}
(c_{nx}+2wc_n)|_A &= \sum_j 2\varphi_{2j}(\varphi_{2jx}+w\varphi_{2j})\sum_{l=0}^{n-1} h_l\lambda_j^{n-l-1}+\tilde{G}_1 \\
&= 2\sum_j \varphi_{2j}^2\sum_{l=0}^{n-1} h_k\lambda_j^{n-l}-2\sum_{m,j}\varphi_{2m}^2\varphi_{1j}\varphi_{2j}\sum_{l=0}^{n-1} h_l\lambda_j^{n-l-1}+\tilde{G}_1
\end{aligned}$$

which, together with (3.26), leads to

$$[v_t-(c_{nx}+2wc_n)]|_A=\tilde{G}_1 . \tag{3.27}$$

In similar manner, we have

$$[u_t-(b_{nx}-2wb_n)]|_A=\tilde{G}_2 . \tag{3.28}$$

Note that

$$\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l(\lambda_j^{n-k}\lambda_m^{k-l}-\lambda_m^{n-k}\lambda_j^{k-l})=\sum_{l=0}^{n-1} h_l(\lambda_m^{n-l}-\lambda_j^{n-l})$$

$$\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l(\lambda_j^{n-k+1}\lambda_m^{k-l-1}-\lambda_m^{n-k+1}\lambda_j^{k-l-1})=\sum_{l=0}^{n} h_l(\lambda_j^{n-l}-\lambda_m^{n-l}).$$

We find from (3.12), (3.14), (3.15), (3.23) and (3.25) that

$$\begin{aligned}
w_t|_A &= \sum_{j,m}\varphi_{2j}^2\varphi_{1m}\varphi_{1mx}\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l\lambda_j^{n-k}\lambda_m^{k-l-1} \\
&\quad -\sum_{j,m}\varphi_{1j}^2\varphi_{2m}^2\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l(\lambda_j^{n-k+1}\lambda_m^{k-l-1}-\lambda_m^{n-k+1}\lambda_j^{k-l-1}) \\
&\quad -\sum_{j,m}\varphi_{1j}\varphi_{2j}(\varphi_{1m}\varphi_{2mx}+\varphi_{1mx}\varphi_{2m})\sum_{k=1}^{n-1}\sum_{l=0}^{k-1} h_l\lambda_j^{n-k}\lambda_m^{k-l-1} \\
&\quad +\sum_{j,m}\varphi_{2j}\varphi_{2jx}\varphi_{1m}^2\sum_{k=1}^{n}\sum_{l=0}^{k-1} h_l\lambda_j^{n-k}\lambda_m^{k-l-1} \\
&\quad +2\sum_{j,m}\varphi_{2m}^2\varphi_{1j}^2\sum_{k=0}^{n-1}\lambda_j^{n-k}h_k-2\sum_{j,m}\varphi_{1j}^2\varphi_{2m}^2\sum_{k=0}^{n-1}\lambda_m^{n-k}h_k+\tilde{G}_3 \\
&= \sum_{j,m}(\varphi_{2j}^2\varphi_{1m}^2-\varphi_{1j}^2\varphi_{2m}^2)\sum_{k=0}^{n-1} h_k\lambda_m^{n-k-1}+\tilde{G}_3
\end{aligned}$$

and

$$a_{nx}|_A=\sum_{j,m}(\varphi_{1j}^2\varphi_{2m}^2-\varphi_{2j}^2\varphi_{1m}^2)\sum_{k=0}^{n-1}h_k\lambda_m^{n-k-1}+\tilde{G}_3.$$

Then

$$(w_t+a_{nx})|_A=\tilde{G}_3. \tag{3.29}$$

By (3.27)-(3.29), (3.16) becomes

$$F_{jt}=\bar{N}_jF_j-E_j \qquad j=1,\ldots,N \tag{3.30}$$

where

$$E_j=\begin{pmatrix}E_{1j}\\E_{2j}\end{pmatrix}=\begin{pmatrix}\tilde{G}_3\varphi_{1j}+\tilde{G}_2\varphi_{2j}\\\tilde{G}_1\varphi_{1j}-\tilde{G}_3\varphi_{2j}\end{pmatrix}.$$

Since $\Phi(x,0)$ solves (3.13), we have

$$F(x,0)\equiv 0$$
$$\left.\frac{\partial^m F}{\partial x^m}\right|_{t=0}\equiv 0 \qquad m=0,1,\ldots. \tag{3.31}$$

Like $\tilde{G}_i$, each monomial of E_{1j} and E_{2j} contains at least one component of F, F_x, $F_{xx},\ldots$ either as a factor or in its integrand, thus one gets from (3.30) and (3.31) that

$$F_t|_{t=0}\equiv 0$$
$$\left.\frac{\partial^{m+1}F}{\partial x^m\partial t}\right|_{t=0}\equiv 0 \qquad m=0,1,\ldots.$$

Finally, it is easy to show by induction that

$$\left.\frac{\partial^{m+l}F}{\partial x^m\partial t^l}\right|_{t=0}\equiv 0 \qquad m,l=0,1,\ldots$$

which yields

$$F(x,t)\equiv 0. \tag{3.32}$$

Then we complete the proof by using (3.15) and (3.16). □

Theorem 3.2. The constants of the motion for (3.13) are given by

$$h_k=\sum_{i=0}^{k}\alpha_i a_{k-i}|_A \qquad k=1,2,\ldots \tag{3.33}$$

with

$$\alpha_0=1 \qquad \alpha_m=-\sum_{l=0}^{m-1}\alpha_l\sum_j\lambda_j^{m-1-l}\varphi_{1j}\varphi_{2j}$$

or

$$h_k=a_k|_A-\sum_j\varphi_{1j}\varphi_{2j}\sum_{m=0}^{k-1}h_m\lambda_j^{k-m-1} \qquad h_0=1 \qquad k=1,2,\ldots. \tag{3.34}$$

Proof. If Φ satisfies (3.13), it is found from (3.23), by using (3.32), that

$$L^{*k}f_0|_A = \left.\begin{pmatrix} c_k \\ b_k \\ 2a_k \end{pmatrix}\right|_A = \sum_j \Psi_j \sum_{m=0}^{k-1} h_m \lambda_j^{k-1-m} + h_k \Psi_0 \tag{3.35}$$

where the h_m are constants defined by

$$h_0 = 1 \qquad h_m = \sum_{i=0}^{m} \alpha_i a_{m-i}|_{A,x=x_0}$$

$$\alpha_0 = 1 \qquad \alpha_m = -\sum_{l=0}^{m-1} \alpha_l \sum_j \lambda_j^{m-1-l} \varphi_{1j}\varphi_{2j}|_{x=x_0}.$$

However, (3.35) implies that the constants h_m only depend on $\lambda_1 \ldots \lambda_N$ and Φ, and have nothing to do with x_0. Thus the h_k given by (3.33) are the constants of the motion for (3.13). Then (3.34) follows from (3.35) immediately. □

Theorem 3.3. If Φ is a solution to (3.13), then p given by (3.12) satisfies a certain higher-order stationary equation

$$\theta L^{*N+1} f_0 + \sum_{k=0}^{N-1} d_k \theta L^{*k+1} f_0 = 0 \tag{3.36}$$

where the d_k are some constants determined by $\lambda_1, \ldots, \lambda_N$ and $h_1, \ldots, h_N$.

Proof. Setting

$$Q(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_N) = \lambda^N + \sum_{k=1}^{N} g_k \lambda^{N-k}$$

then

$$Q(\lambda_j) = \lambda_j^N + \sum_{k=1}^{N} g_k \lambda_j^{N-k} = 0. \tag{3.37}$$

Using (3.35) gives

$$\begin{aligned} \sum_{k=0}^{N} d_k L^{*k+1} f_0|_A &= \sum_j \Psi_j \sum_{k=0}^{N} d_k \sum_{m=0}^{k} h_m \lambda_j^{k-m} + \Psi_0 \sum_{k=0}^{N} d_k h_{k+1} \\ &= \sum_j \Psi_j \sum_{k=0}^{N} \lambda_j^{N-k} \sum_{m=0}^{k} h_m d_{N-k-m} + \Psi_0 \sum_{k=0}^{N} d_k h_{k+1}. \end{aligned} \tag{3.38}$$

Taking $d_N = 1$ and

$$\sum_{m=0}^{k} h_m d_{N-k-m} = g_k \qquad k = 1, \ldots, N$$

or

$$d_{N-k} = g_k - \sum_{m=1}^{k} h_m d_{N-k-m} \qquad k = 1, \ldots, N$$

then

$$\sum_{k=0}^{N} d_k L^{*k+1} f_0 = \Psi_0 \sum_{k=0}^{N} d_k h_{k+1}$$

which leads to (3.36) immediately. □

For (1.3b), the time evolution equation of Φ which corresponds to (3.14) is [9]

$$\Phi_{jt} = \bar{N}_j \Phi_j \qquad \bar{N}_j = \begin{pmatrix} \bar{A}_j & \bar{B}_j \\ \bar{C}_j & -\bar{A}_j \end{pmatrix} \tag{3.39}$$

where

$$\bar{A}_j = \left(\sum_{k=0}^{n} a_k \lambda_j^{n-k} + l a_{n+1} \right) |_A$$

$$\bar{B}_j = \sum_{k=1}^{n} b_k \lambda_j^{n-k} |_A \qquad \bar{C}_j = \sum_{k=1}^{n} c_k \lambda_j^{n-k} |_A$$

$$\begin{pmatrix} c_k \\ b_k \\ 2a_k \end{pmatrix} = L^{*k} f_0 \qquad k = 0, 1, \ldots, n+1.$$

Then similar results are valid for (1.3b).

Theorem 3.4. If $\Phi(x, t)$ solves (3.39) and $\Phi(x, 0)$ solves (3.13), then $\Phi(x, t)$ satisfies (3.13) and p given by (3.12) is a solution to (1.3b).

The proof can be done in same way as we did for theorem 3.1.

Theorem 3.5. If Φ satisfies (3.13), then p given by (3.12) is a solution of a certain higher-order stationary equation

$$\theta_l L^{*N+1} f_0 + \sum_{k=0}^{N-1} d_k \theta_l L^{*k+1} f_0 = 0. \tag{3.40}$$

Proof. Using (3.37) and (3.38) and taking

$$d_N = 1$$

$$d_{N-1} = g_k - \sum_{m=1}^{k} h_m d_{N-k+m} \qquad k = 1, \ldots, N-1$$

$$(1 + l h_1) d_0 = g_N - \sum_{m=1}^{N} d_m (h_m + l h_{m+1})$$

we obtain (3.40). □

Finally, for the Yang equation (1.2), we can get the following natural constraint on potential $\bar{p}$ by constructing an invariant subspace of $\bar{L}^*$

$$q = -4\left(\sum_j \psi_{1j}\psi_{2j} \right)^2 - \left(\sum_j (\psi_{2j}^2 - \psi_{1j}^2) \right)^2 - \sum_j \xi_j (\psi_{2j}^2 + \psi_{1j}^2) + \sum_j (\psi_{1jx}\psi_{2j} - \psi_{1j}\psi_{2jx})$$

$$r = \sum_j (\psi_{2j}^2 - \psi_{1j}^2) \qquad s = 2 \sum_j \psi_{1j}\psi_{2j} \qquad \sum_j (\psi_{1j}^2 + \psi_{2j}^2) = 1$$

which can also be obtained from (3.11c) and (3.12) by using the gauge transformation relating (1.1) and (1.2) given in [9].

Then similar theorems to theorems 3.1–3.3 hold for the Yang equation (1.2) and the evolution equations (1.4).

Some other algebraic geometrical properties associated with this non-hereditary operator, such as the Lie algebra structure of the symmetries, will be discussed in a forthcoming paper.

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